THE METHOD OF DARBOUX

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THE METHOD OF DARBOUX

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ABSTRACT

The problem of approximating the terms of a sequence was investigated by M. G. Darboux in 1878. Since then the theory of asymptotic expansions has been greatly extended. The purpose of this thesis is to obtain the Darboux result by more modern means and in a form more consistent with present theory.

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INTRODUCTION

The concept of asymptotic expansions originated with H. Poincaré in 1886. The usefulness of these expansions has resulted in numerous generalizations of this initial definition. One of the most recent and elegant of these definitions is due to A. Erdélyi and is discussed in chapter one. A specialization of this definition is adopted for the purpose of the thesis and some relevant theorems are established.

The thesis is concerned with the asymptotic behavior of sequences whose generating function has a finite number of poles or branch points on the circle of convergence. A method of approximating the terms of such a sequence was developed by M. G. Darboux in 1878. His proof depended on the order of magnitude of the terms in a trigonometric series. Our proof of the Darboux result is presented in chapter two. This proof is obtained by means of contour integration. It yields the final result in a more convenient form and is, as far as we know, unpublished.

In spite of the convenience of this method and the large number of sequences to which it can be applied, it does not seem to be well known. Some of the examples given in chapter three deal with sequences whose asymptotic behavior has been established by some other method, and to which the method of Darboux is applicable. In all of these cases, the method used is specialized and yields only the dominant term of the expansion. On the other hand, the method of Darboux gives the complete asymptotic behavior of each sequence easily.

CHAPTER I

DEFINITIONS OF ASYMPTOTIC EXPANSIONS

The idea of an asymptotic expansion of a function began with Poincaré in 1886 [1.1] . His definition is as follows.

DEFINITION 1.1 :

A series

$$(1.1) \qquad A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \dots + \frac{A_n}{z^n} + \dots$$

is called an asymptotic expansion of the function $f(z)$, in a given range of arg z, if, for every fixed value of n ,

$$(1.2) \qquad \lim_{|z| \to \infty} z^n \left\{ f(z) - A_0 - \frac{A_1}{z} - \dots - \frac{A_n}{z^n} \right\} = 0 .$$

The notation used to denote such a relationship between the series and $f(z)$ is

$$(1.3) \qquad f(z) \sim A_0 + \frac{A_1}{z} + \dots + \frac{A_n}{z^n} + \dots \quad .$$

The following properties of an asymptotic expansion are well known.

If a function $f(z)$ possesses an asymptotic expansion in Poincaré's sense, this expansion is unique. These unique coefficients can be determined successively by the equations

$$(1.4) \qquad A_0 = \lim_{|z| \to \infty} f(z)$$

$$(1.5) \qquad A_1 = \lim_{|z| \to \infty} z \left\{ f(z) - A_0 \right\}$$

$$(1.6) \qquad A_n = \lim_{|z| \to \infty} z^n \left\{ f(z) - A_0 - \frac{A_1}{z} - \dots - \frac{A_{n-1}}{z^{n-1}} \right\}$$

If

$$(1.7) \qquad f(z) \sim \sum_{r=0}^{\infty} A_r \, z^{-r}$$

and

$$(1.8) \qquad g(z) \sim \sum_{r=0}^{\infty} B_r \, z^{-r}$$

for the same range of values of $\arg z$, then

$$(1.9) \qquad \alpha\, f(z) + \beta\, g(z) \sim \sum_{r=0}^{\infty} (\alpha\, A_r + \beta\, B_r)\, z^{-r} ,$$

and

$$(1.10) \qquad f(z)\; g(z) \sim \sum_{r=0}^{\infty} C_r \, z^{-r} ,$$

where

$$(1.11) \qquad C_r = A_0 B_r + A_1 B_{r-1} + \dots + A_r B_0 .$$

If

$$(1.12) \qquad f(z) \sim \sum_{r=0}^{\infty} A_r \, z^{-r} , \qquad \alpha < \arg z < \beta$$

then, $\quad f(z) - A_0 - \dfrac{A_1}{z} \quad$ is integrable, and

$$(1.13) \qquad F(z) = \int_z^{\infty} \left\{ f(t) - A_0 - \frac{A_1}{t} \right\} dt$$

$$\sim \frac{A_2}{z} + \frac{A_3}{2z^2} + \frac{A_4}{3z^3} + \dots , \qquad \alpha < \arg z < \beta .$$

Let $f(z)$ be a differentiable function. Then if

$$(1.14) \qquad f(z) \sim \sum_{r=0}^{\infty} A_r z^{-r} \quad , \quad \alpha < \arg z < \beta$$

and if $f'(z)$ possesses an asymptotic expansion, then

$$(1.15) \qquad f'(z) \sim -\underline{\frac{A_1}{z^2} - \frac{2A_2}{z^3}} - \dots , \quad \alpha < \arg z < \beta \ .$$

It should be noted that nothing has been said about the convergence of the asymptotic series $\sum_{r=0}^{\infty} A_r z^{-r}$. From the point of view of usefulness, the convergence or divergence is unimportant.

If a function $f(z)$ does possess an asymptotic expansion, this expansion may be used to compute the values of $f(z)$ for large values of z , the error after n terms being $o(1/z^{n-1})$. In fact, in most cases, the first few terms are sufficient for computational purposes, the accuracy increasing as $|z|$ increases.

It is easily shown that there exist functions that do not possess asymptotic expansions in the Poincaré sense. For example,

$$(1.16) \qquad z^{\frac{1}{2}}/(z^{\frac{1}{2}} + 1) = 1 - z^{-\frac{1}{2}} + z^{-1} - z^{-\frac{3}{2}} + \dots$$

$$(1.17) \qquad (z + \log z)^{-1} = z^{-1} \left\{ 1 - (\log z)/z + (\log z)^2/z^2 - \dots \right\} .$$

In the first expansion the powers of z are not all integers; in the second, the factor z^{-1} multiplies the entire series and the numerators of the terms of the series are functions of z rather than constants. Yet each of these series has many of the desirable properties of an asymptotic expansion.

The usefulness of asymptotic expansions has resulted in numerous generalizations of Poincaré's concept.

One of the most recent and most elegant definitions of an asymptotic expansion is due to Professor A. Erdélyi. This definition has not yet been published and was disclosed in a private conversation. It is dependent on the concept of an asymptotic sequence which was introduced by Erdélyi [1.2] .

In any discussion to follow, the complex variable z will lie in a region R and z_o will be a limit point of R, which may or may not lie in R .

DEFINITION 1.2 :

The finite or infinite sequence of functions $\{\phi_n(z)\}$ is called an asymptotic sequence as $z \to z_o$ in a region R if for each n , $\phi_n(z)$ is defined in R and $\phi_{n+1}(z) = o(\phi_n(z))$ as $z \to z_o$ in R.

Among the examples which Erdélyi gives of asymptotic sequences are the following:

(i) $\{ z^{-\lambda_n} \}$, where $\mathrm{Rl}\ \lambda_{n+1} > \mathrm{Rl}\ \lambda_n$ for each n , and $z \to \infty$ in the sector $0 < |z| < \infty$, $|\arg z| < \pi$.

(ii) $\{ e^{-nz} z^{-\lambda_n} \}$, under the same conditions as above, except $|\arg z| \leq \frac{\pi}{2}$.

Two sequences, $\{\phi_n(z)\}$, $\{\psi_n(z)\}$ such that $\phi_n(z) = O(\psi_n(z))$ and $\psi_n(z) = O(\phi_n(z))$, are said to be equivalent. It is easily shown that if $\{\phi_n(z)\}$ and $\{\psi_n(z)\}$

are equivalent sequences and $\{\phi_n(z)\}$ is an asymptotic sequence, then $\{\psi_n(z)\}$ is also an asymptotic sequence.

Erdélyi also discusses various methods of obtaining, from given asymptotic sequences, new such sequences. For example,

(i) Any subsequence of an asymptotic sequence is an asymptotic sequence.

(ii) If $\{\phi_n(z)\}$ is an asymptotic sequence and $\alpha > 0$, then $\{|\phi_n(z)|^\alpha\}$ is an asymptotic sequence.

(iii) If $\{\phi_n(z)\}$ and $\{\psi_n(z)\}$ are asymptotic sequences containing the same number of functions, then $\{\phi_n(z) \quad \psi_n(z)\}$ is an asymptotic sequence.

We are now in a position to state Erdélyi's definition.

DEFINITION 1.3 :

Let $\{\phi_r(z)\}$ be an asymptotic sequence as $z \to z_o$ in R. Then the series $\Sigma\ \psi_r(z)$ is said to be an asymptotic expansion to N terms of $f(z)$ as $z \to z_o$ in R, if

$$(1.18) \qquad f(z) = \sum_{r=1}^{s} \psi_r(z) + o(\phi_s(z)) \quad \text{as} \quad z \to z_o\,, \quad s = 1, \ldots, N\,.$$

An immediate consequence of this definition is

$$(1.19) \qquad \psi_{r+1}(z) = o(\phi_r(z)) \qquad \text{as} \quad z \to z_o\,, \quad r = 1, \ldots, N-1\,.$$

We use the notation

$$(1.20) \qquad f(z) \sim \sum_{r=1}^{N} \psi_r(z)$$

to denote this relationship.

If we let $\psi_r(z) = A_r \phi_r(z)$, where the A_r's are constants, we obtain Erdélyi's published definition.

If a function $f(z)$ possesses an asymptotic expansion in terms of a particular asymptotic sequence, this expansion is unique. On the other hand, one and the same function may have asymptotic expansions involving two different asymptotic sequences, and the two sequences need not be equivalent. An asymptotic expansion does not determine the corresponding function uniquely, since several functions may possess the same asymptotic expansion. Therefore, it seems desirable to place such functions into the same equivalence class. Erdélyi accomplishes this by the following definition of asymptotic equality [1.2] .

DEFINITION 1.4 :

Let $f(z)$ and $g(z)$ be two functions defined in R and let $\{\phi_r(z)\}$ be an asymptotic sequence for $z \to z_0$ in R. Then $f(z)$ and $g(z)$ are said to be asymptotically equal with respect to $\{\phi_r(z)\}$ if

$$(1.21) \qquad f(z) - g(z) = o(\phi_r(z)) \quad \text{as} \quad z \to z_0 \quad \text{in } R, \text{ for each } r.$$

Clearly, if $f(z) \sim \Sigma A_r \phi_r(z)$ as $z \to z_0$ and if $f(z)$ is asymptotically equal to $g(z)$ with respect to $\{\phi_r(z)\}$, then $g(z) \sim \Sigma A_r \phi_r(z)$ as $z \to z_0$.

The relationship of asymptotic equality between functions is obviously a true equivalence relation.

Thus each asymptotic sequence separates all functions into classes of asymptotically equal functions in such a way that all members of a particular class possess identical asymptotic expansions.

The dependence of these asymptotic expansions on asymptotic sequences can be inconvenient. For example, if $f(z) \sim \Sigma\ A_r\ \phi_r$, the identical series

$$A_1 \phi_1 + 0 + A_2 \phi_2 + 0 + A_3 \phi_3 + 0 + \dots , \tag{1.22}$$

is no longer an asymptotic expansion of $f(z)$ since the sequence

$$\phi_1 , 0 , \phi_2 , 0 , \phi_3 , 0 , \dots \tag{1.23}$$

is not an asymptotic sequence. Since this objection is overcome by Erdélyi's more general definition, we will use the following specialization of this latter definition for the remainder of this thesis.

Let $\{A_r(z)\}$ be a sequence of functions defined in R , and let $U(z)$ be a function such that $|U(z)| \to \infty$ as $z \to z_o$ in R. Clearly, $\{(U(z))^{-r}\}$ is an asymptotic sequence. The formal sum $\sum_{r=o}^{\infty} A_r(z)/(U(z))^r$ is said to be an asymptotic expansion of $f(z)$ as $z \to z_o$ if, for all n ,

$$f(z) - \sum_{r=o}^{n} A_r(z)/(U(z))^r = o(1/[U(z)]^n) \text{ as } z \to z_o . \tag{1.24}$$

Again we use the notation

$$(1.25) \quad f(z) \sim \sum_{r=0}^{\infty} A_r(z)/(U(z))^r \quad .$$

The usual meaning is attached to

$$(1.26) \quad f(z) \sim g(z) \quad \sum_{r=0}^{\infty} A_r(z)/(U(z))^r \quad .$$

THEOREM 1.1 :

If $f(z) \sim \sum_{r=0}^{\infty} A_r(z)/(U(z))^r$ as $z \to z_0$ in R ,

then $\lim_{z \to z_0} A_n(z)/U(z) = 0$, for all n .

PROOF:

As a result of our definition, we have

$$(1.27) \quad f(z) - \sum_{r=0}^{n} A_r(z)/(U(z))^r = o(1/[U(z)]^n)$$

and

$$(1.28) \quad f(z) - \sum_{r=0}^{n+1} A_r(z)/(U(z))^r = o(1/[U(z)]^{n+1}) \quad \text{as} \quad z \to z_0 \; .$$

Hence,

$$(1.29) \quad A_{n+1}(z)/(U(z))^{n+1} = o(1/[U(z)]^n)$$

and

$$(1.30) \qquad A_{n+1}(z)/U(z) = o(1) \quad ,$$

for all n, i.e.,

$$(1.31) \qquad \lim_{z \to z_0} A_n(z)/U(z) = 0 \quad .$$

For convenience we shall adopt the following specialization of the Erdélyi definition of asymptotic equivalence.

Two functions $f(z)$ and $g(z)$ will be called asymptotically equivalent with respect to $U(z)$ at $z = z_0$, if positive constants a and b, and a neighborhood N of z_0 exist such that, for all z common to N and R,

$$(1.32) \qquad \left| f(z) - g(z) \right| \leq a\, e^{-b|U|} \quad .$$

For example, the functions $f(z) = z$ and $g(z) = z + z^{-\frac{1}{2}}$ are asymptotically equivalent with respect to $U(z) = \ln z$ at $z = \infty$. However, the same two functions are not asymptotically equivalent with respect to $U(z) = z$ at $z = \infty$.

This again is obviously an equivalence relation.

THEOREM 1.2 :

A necessary and sufficient condition for two functions $f(z)$ and $g(z)$ to be asymptotically equivalent with respect to $U(z)$ at z_0, is that there exist a sequence of positive numbers $\{a_n\}$, a positive number "c", and a fixed neighborhood of z_0 for which

(i) $\sum_{n=0}^{\infty} a_n c^n$ converges

(ii) $|U|^n \left| f(z) - g(z) \right| \leq a_n n!$ for all z in the intersection of R and the neighborhood.

PROOF:

Let us first prove the necessity. There exist constants a and b, and a neighborhood N of z_0 such that, for z common to N and R,

$$(1.33) \qquad \left| f(z) - g(z) \right| \leq a\, e^{-b|U|} .$$

Therefore,

$$(1.34) \qquad |U|^n \left| f(z) - g(z) \right| \leq a\, |U|^n\, e^{-b|U|} .$$

If we consider, for a moment, the function $w = x^n\, e^{-bx}$, $x > 0$, we see that since

$$(1.35) \qquad \frac{dw}{dx} = x^{n-1}\, e^{-bx}\, (n - bx) ,$$

w must have a horizontal tangent at $x = n/b$. This point obviously yields a maximum since w is a positive, continuous function which tends to zero as x tends to ∞ or 0.

Applying this, we obtain

$$(1.36) \qquad |U|^n \left| f(z) - g(z) \right| \leq a \left(\frac{n}{b}\right)^n e^{-n}$$

$$= \frac{a\sqrt{2\pi}\; n^{n+\frac{1}{2}}\; e^{-n}}{\sqrt{2\pi}\; n^{\frac{1}{2}}\; b^n}$$

$$\leq \frac{a}{\sqrt{2\pi n}\; b^n}\, n! .$$

If we choose $a_n = \dfrac{a}{\sqrt{2\pi n}\; b^n}$, condition (ii) of the theorem is satisfied. By a simple application of the ratio test,

$$(1.37) \qquad \sum_{n=1}^{\infty} a_n c^n = \frac{a}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{c^n}{n^{\frac{1}{2}} b^n}$$

is seen to be convergent for $c < b$. Thus condition (i) is also satisfied.

To prove the sufficiency, we deduce from

$$(1.38) \qquad |U|^n \left| f(z) - g(z) \right| \leq a_n n!$$

that

$$(1.39) \qquad \sum_{n=0}^{\infty} \frac{|c\, U|^n}{n!} \left| f(z) - g(z) \right| \leq \sum_{n=0}^{\infty} a_n c^n \leq a$$

and

$$(1.40) \qquad \left| f(z) - g(z) \right| \leq a\, e^{-c|U|} \quad .$$

Thus, from the definition of asymptotic equivalence, it follows that $f(z)$ and $g(z)$ are asymptotically equivalent with respect to U at z_o.

The following two theorems connect the concept of asymptotic equivalence to that of asymptotic expansions.

THEOREM 1.3 :

If the functions $f(z)$ and $g(z)$ are asymptotically equivalent with respect to $U(z)$ at $z = z_o$, then

$$f(z) \sim g(z) \quad \text{as} \quad z \to z_o \quad .$$

PROOF:

This follows trivially since there exist constants a and b and a neighborhood N of z_o such that, for all z common to N and R,

(1.41) $$\left| f(z) - g(z) \right| \leq a\, e^{-b|U|} = o(1/U^n) \text{ for all } n, \text{ as } z \to z_o .$$

THEOREM 1.4 :

If $f(z) \sim \sum_{r=o}^{\infty} A_r(z)/(U(z))^r$ as $z \to z_o$, and if the corresponding terms of the two sequences $\{A_r(z)\}$ and $\{B_r(z)\}$ are asymptotically equivalent with respect to $U(z)$ at $z = z_o$, then $f(z) \sim \sum_{r=o}^{\infty} B_r(z)/(U(z))^r$ as $z \to z_o$.

PROOF:

As a result of the preceding theorem,

(1.42) $$A_n(z) - B_n(z) = o(1/U^n) \text{ for all } n, \text{ as } z \to z_o .$$

Consequently,

(1.43)
$$\begin{aligned} & f(z) - \sum_{r=o}^{n} B_r(z)/(U(z))^r \\ &= f(z) - \sum_{r=o}^{n} \left\{ A_r(z) + \left[B_r(z) - A_r(z) \right] \right\} /(U(z))^r \\ &= f(z) - \sum_{r=o}^{n} A_r(z)/(U(z))^r + o(1/U^n) \\ &= o(1/U^n) \quad \text{as } z \to z_o . \end{aligned}$$

Therefore,

(1.44) $$f(z) \sim \sum_{r=o}^{\infty} B_r(z)/(U(z))^r \quad \text{as } z \to z_o .$$

CHAPTER II

THE METHOD OF DARBOUX

In the year 1878 Darboux considered the problem of determining the asymptotic behavior of a sequence of numbers. [2.1] In our present chapter we shall give an alternate derivation of his result under slightly different assumptions.

We shall assume that a generating function $f(z)$ exists for the sequence $\{a_n\}$ of the form

$$(2.1)\qquad f(z) = \sum_{n=0}^{\infty} a_n z^n \quad ,$$

and that this generating function has the following properties:

(A) $f(z)$ is a regular function of z for $|z| < R$.

(B) $f(z)$ has a finite number of singularities on its circle of convergence at $z = \alpha_j$, $j = 1,\ldots,m$.

(C) In the neighborhood of the singularity $z = \alpha_j$, $f(z)$ has the form

$$(2.2)\qquad f(z) = (z - \alpha_j)^{-p_j} g_j(z) + h_j(z)$$

where $g_j(z)$ and $h_j(z)$ are regular functions of z in a neighborhood of $z = \alpha_j$, and $\mathrm{Rl}\ p_j > 0$, $j = 1,\ldots,m$.

Our assumptions imply that the singularities of $f(z)$ which lie on the circle of convergence are either poles or branch points.

By our assumptions, $g_j(z)$ has a Taylor expansion about $z = \alpha_j$ of the form

$$(2.3) \qquad g_j(z) = \sum_{r=o}^{\infty} g_j^{(r)}(\alpha_j)(z - \alpha_j)^r / r! \quad ,$$

which we write as

$$(2.4) \qquad g_j(z) = \sum_{r=o}^{k-1} g_j^{(r)}(\alpha_j)(z - \alpha_j)^r / r! + (z - \alpha_j)^k K_j(z) \quad .$$

Let $s = \max(\mathrm{Rl}\ p_j)$, $j = 1,\ldots,m$. Since k is arbitrary, we may choose $k > [s]$, where $[s]$ denotes the largest integer contained in s. Further, we define an integer q by

$$(2.5) \qquad q = k - [s] \quad .$$

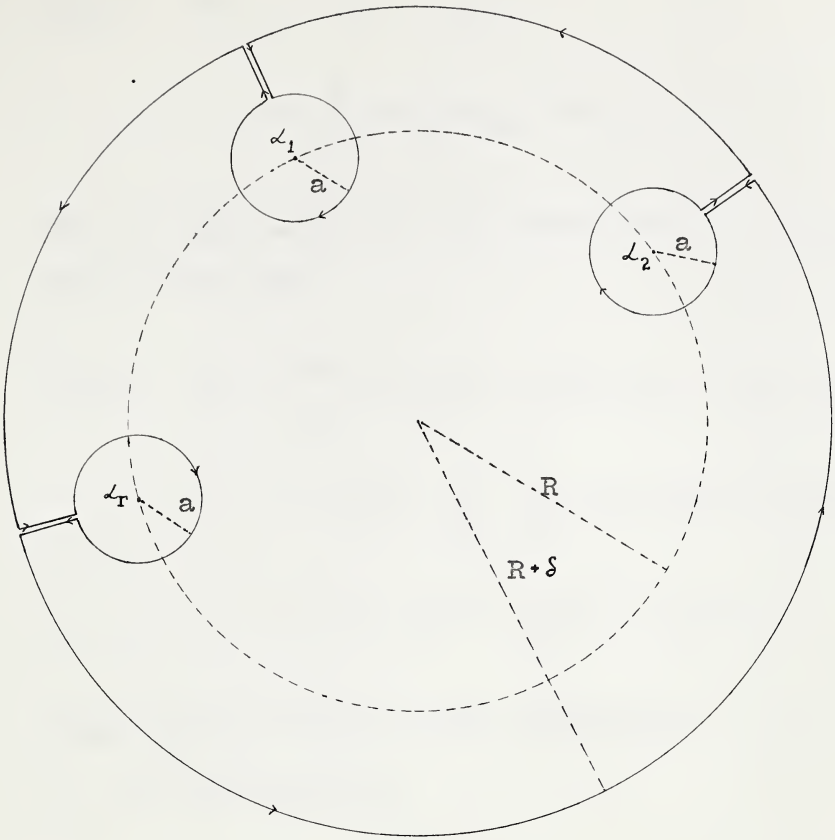
From (2.4), we can write

$$(2.6) \qquad f(z) = \sum_{r=o}^{k-1} g_j^{(r)}(\alpha_j)(z - \alpha_j)^{r-p_j} / r!$$

$$+ (z - \alpha_j)^{k-p_j} K_j(z) + h_j(z) \quad ,$$

in a neighborhood of $z = \alpha_j$.

If we make appropriate cuts in the z plane, it is possible to find a region of the shape shown below, within which $f(z)$ is regular. This region shall be denoted by $U(a , \delta)$, or simply U . If desired, U could be made to extend out to, but not including, the next singularity of $f(z)$. However, in our proof, both "a" and "δ" will eventually be made to approach zero.



The expression $D_k(z)$, given by

$$(2.7) \qquad D_k(z) = \sum_{j=1}^{m} \sum_{r=o}^{k-1} g_j^{(r)}(\alpha_j)(z-\alpha_j)^{r-p_j}/r! \quad ,$$

is called the Darboux approximant of $f(z)$. Clearly, $D_k(z)$ is regular within the region U, and has a Maclaurin expansion convergent within $|z| < R$. Thus

$$(2.8) \qquad D_k(z) = \sum_{n=o}^{\infty} a_{nk} z^n \quad .$$

The function $F(z)$, defined by

$$(2.9) \qquad F(z) = f(z) - D_k(z)$$

$$= \sum_{n=0}^{\infty} (a_n - a_{nk}) \; z^n ,$$

is also regular within U. In a neighborhood of $z = \alpha_j$, $F(z)$ has the form

$$(2.10) \qquad F(z) = \sum_{r=0}^{k-1} g_j^{(r)}(\alpha_j)(z - \alpha_j)^{r-p_j}/r! + (z - \alpha_j)^{k-p_j} K_j(z)$$

$$+ h_j(z) - \sum_{j=1}^{m} \sum_{r=0}^{k-1} g_j^{(r)}(\alpha_j)(z - \alpha_j)^{r-p_j}/r!$$

$$= (z - \alpha_j)^{k-p_j} K_j(z) + H_j(z) ,$$

where $H_j(z)$ is regular in this neighborhood.

It thus follows that

$$(2.11) \qquad F^{(q)}(z) = (z - \alpha_j)^{k-p_j-q} K_{j,q}(z) + H_j^{(q)}(z)$$

$$= (z - \alpha_j)^{[s]-p_j} K_{j,q}(z) + H_j^{(q)}(z)$$

where the functions $K_{j,q}(z)$ are regular. Thus $F(z)$ has $q-1$ bounded derivatives in a neighborhood of $z = \alpha_j$.

From (2.9), we have

$$(2.12) \qquad a_n - a_{nk} = (2 \pi i)^{-1} \int_C F(z) \, z^{-(n+1)} \, dz ,$$

where C is any contour within the domain of regularity of $F(z)$. Integration by parts q times yields

$$(2.13) \qquad a_n - a_{nk} = \frac{1}{2\pi i} \frac{(n-q)!}{n!} \int_C \frac{F^{(q)}(z)}{z^{n-q+1}} dz .$$

Let us divide the contour C up into three parts:

I The small circles enclosing the singularities.

II The radial lines.

III The outer circle of radius $R + \delta$.

We shall now choose $a = R/n$, and $\delta = R/\sqrt{n}$.

On the small circle enclosing the singularity $z = \alpha_j$, we have

$$(2.14) \qquad z = \alpha_j + a e^{i\theta} .$$

Therefore, from (2.11) constants M and N exist such that

$$(2.15) \qquad \left| F^{(q)}(z) \right| \leq a^{[s]-Rl\, p_j} M + N ,$$

and

$$(2.16) \qquad \left| \int_I \frac{F^{(q)}(z)}{z^{n-q+1}} dz \right| \leq \frac{a^{[s]-Rl\, p_j+1} M + a N}{(R - a)^{n-q+1}} 2\pi$$

$$= o\left(\frac{1}{R^n}\right) .$$

On the radial lines,

$$(2.18) \qquad z = r e^{i \arg \alpha_j} .$$

Again from (2.11), we have

$$(2.19) \qquad \left| F^{(q)}(z) \right| \leq (r - R)^{[s]-Rl\, p_j} M_1 + N_1 .$$

Thus,

$$(2.20)\quad \left| \int_{II} \frac{F^{(q)}(z)}{z^{n-q-1}}\, dz \right| \leq \int_{R+a}^{R+\delta} \frac{(r-R)^{[s]-Rl\, p_j} M_1 + N_1}{R^{n-q+1}}\, dr$$

$$= \frac{\left(\delta^{[s]-Rl\, p_j+1} - a^{[s]-Rl\, p_j+1}\right) M_1' + N_1 (\delta - a)}{R^{n-q+1}}$$

$$= o\left(\frac{1}{R^n}\right) .$$

On the large circle of radius $R + \delta$,

$$(2.21)\quad z = (R + \delta)\, e^{i\theta} = R\left(1 + \frac{1}{\sqrt{n}}\right) e^{i\theta} .$$

Thus,

$$(2.22)\quad z - \alpha_j = R \left\{ \left(1 + \frac{1}{\sqrt{n}}\right) e^{i\theta} - e^{i \arg \alpha_j} \right\} ,$$

and

$$(2.23)\quad \frac{R}{\sqrt{n}} \leq \left| z - \alpha_j \right| \leq R\left(2 + \frac{1}{\sqrt{n}}\right) .$$

If $[s] - Rl\, p_j > 0$,

$$(2.24)\quad \left| F^{(q)}(z) \right| \leq \left\{ R\left(2 + \frac{1}{\sqrt{n}}\right) \right\}^{[s]-Rl\, p_j} M_2 + N_2 ,$$

and

$$(2.25)\quad \left| \int_{III} \frac{F^{(q)}(z)}{z^{n-q+1}}\, dz \right| \leq \frac{\left\{ R\left(2 + \frac{1}{\sqrt{n}}\right) \right\}^{[s]-Rl\, p_j} M_2 + N_2}{R^{n-q} \left(1 + \frac{1}{\sqrt{n}}\right)^{n-q}}\, 2\pi$$

$$= o\left(\frac{1}{R^n}\right) .$$

If $[s] - \mathrm{Rl}\, p_j < 0$,

$$(2.26) \quad \left| F^{(q)}(z) \right| \leq \left(\frac{R}{\sqrt{n}}\right)^{[s] - \mathrm{Rl}\, p_j} M_2 + N_2 ,$$

and

$$(2.27) \quad \left| \int_{III} \frac{F^{(q)}(z)\, dz}{z^{n-q+1}} \right| \leq \frac{\left(\frac{R}{\sqrt{n}}\right)^{[s] - \mathrm{Rl}\, p_j + 1} M_2 + \frac{R}{\sqrt{n}} N_2}{R^{n-q+1} \left\{ \frac{1}{\sqrt{n}} \left(1 + \frac{1}{\sqrt{n}} \right)^{n-q} \right\}} \, 2\pi$$

$$= o\left(\frac{1}{R^n}\right) .$$

These results and (2.13) show that

$$(2.28) \quad a_n - a_{nk} = \frac{(n-q)!}{n!} \, o\left(\frac{1}{R^n}\right) .$$

Using (2.7) and (2.8) , we obtain

$$(2.29) \quad a_{nk} = \frac{1}{n!} \left[\frac{d^n}{dz^n} D_k(z) \right]_{z=0}$$

$$= \frac{(-1)^n}{n!} \sum_{j=1}^{m} \sum_{r=o}^{k-1} g_j^{(r)}(\alpha_j) (-\alpha_j)^{r-p_j-n} \frac{\Gamma(n+p_j-r)}{r!\,\Gamma(p_j-r)} .$$

Thus

$$(2.30) \quad a_n = \frac{(-1)^n}{n!} \sum_{j=1}^{m} \sum_{r=o}^{k-1} g_j^{(r)}(\alpha_j)(-\alpha_j)^{r-p_j-n} \frac{\Gamma(n+p_j-r)}{r!\,\Gamma(p_j-r)} + \frac{(n-q)!}{n!} \cdot o\left(\frac{1}{R^n}\right)$$

This provides us with an asymptotic expansion for a_n , since for each $k > [s]$,

$$(2.31) \quad \frac{n!\, R^n}{\Gamma(n+s)} a_n = (-1)^n \sum_{j=1}^{m} \sum_{r=o}^{k-1} \frac{g_j^{(r)}(\alpha_j)(-\alpha_j)^{r-p_j}}{r!\,\Gamma(p_j-r)} \left(\frac{R}{-\alpha_j}\right)^n \frac{\Gamma(n+p_j-r)}{\Gamma(n+s)}$$

$$+ \frac{(n-q)!}{\Gamma(n+s)} \cdot o\,(1)$$

which can be put into the form

$$(2.32)\qquad \frac{n!\ R^n}{\Gamma(n+s)}\, a_n = A_o(n) + \frac{A_1(n)}{n} + \dots + \frac{A_{k-1}(n)}{n^{k-1}} + o\left(\frac{1}{n^{k-1}}\right) .$$

Hence

$$(2.33)\qquad a_n \sim \frac{(-1)^n}{n!} \sum_{j=1}^{m} \sum_{r=o}^{\infty} g_j^{(r)}(\alpha_j)(-\alpha_j)^{r-p_j-n} \frac{\Gamma(n+p_j-r)}{r!\,\Gamma(p_j-r)} .$$

In the event that α_i is a pole of order p_i, it is easily seen that its contribution to a_{nk} and, therefore also to the asymptotic expansion of a_n, contains only p_i terms.

CHAPTER III

APPLICATIONS

As a first example, we shall obtain the asymptotic expansion of Legendre polynomials of large degree. The generating function of the Legendre polynomials is

$$(3.1) \qquad \left\{ 1 - 2tz + z^2 \right\}^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(t)\, z^n \quad .$$

We shall first choose $t = \cosh x$, where x is real and positive. Thus, (3.1) becomes

$$(3.2) \qquad \sum_{n=0}^{\infty} P_n(\cosh x)\, z^n = \left\{ 1 - (e^x + e^{-x})\, z + z^2 \right\}^{-\frac{1}{2}}$$
$$= -\left\{ z - e^{-x} \right\}^{-\frac{1}{2}} \left\{ z - e^x \right\}^{-\frac{1}{2}} \quad .$$

The minus sign comes from choosing the branches of the functions $\left\{ z - e^{-x} \right\}^{-\frac{1}{2}}$ and $\left\{ z - e^{x} \right\}^{-\frac{1}{2}}$ that are equal to $-i\, e^{\frac{1}{2}x}$ and $-i\, e^{-\frac{1}{2}x}$ respectively at $z = 0$.

Clearly, the first singularity is at $z = e^{-x}$. Therefore, in the notation of the preceding chapter,

$$(3.3) \qquad g(z) = -\left\{ z - e^x \right\}^{-\frac{1}{2}} \quad ,$$

$$(3.4) \qquad g^{(r)}(z) = \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})} (-1)^{r+1} (z - e^x)^{-(r+\frac{1}{2})} \quad ,$$

and

$$(3.5) \qquad g^{(r)}(e^{-x}) = \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})} (-1)^{r+1} \left(-2 \sinh x \right)^{-(r+\frac{1}{2})} \quad .$$

From (2.33), we have

$$(3.6)\quad P_n(\cosh x) \sim \frac{e^{x(\frac{1}{2}+n)}}{n!\,\Gamma(\frac{1}{2})} \sum_{r=o}^{\infty} \frac{\Gamma(r+\frac{1}{2})\,\Gamma(n-r+\frac{1}{2})}{\Gamma(\frac{1}{2}-r)\quad r!} (-1)^r\, e^{-xr}\,(2\sinh x)^{-(r+\frac{1}{2})}.$$

As a result of the identity $\Gamma(w)\,\Gamma(1-w) = \pi/\sin(\pi w)$,

$$(3.7)\quad \Gamma(\tfrac{1}{2}-r)\,\Gamma(\tfrac{1}{2}+r) = \pi(-1)^r , \quad \text{and} \quad \Gamma(n-r+\tfrac{1}{2})\,\Gamma(r-n+\tfrac{1}{2}) = \pi(-1)^{n-r} .$$

Thus

$$(3.8)\quad P_n(\cosh x) \sim \frac{e^{x(\frac{1}{2}+n)}}{\sqrt{2\pi\sinh x}} \frac{(-1)^n}{n!} \sum_{r=o}^{\infty} \frac{\Gamma(r+\frac{1}{2})\,\Gamma(r+\frac{1}{2})}{r!\,\Gamma(r-n+\frac{1}{2})} \left\{\frac{-e^{-x}}{2\sinh x}\right\}^r$$

$$= \frac{e^{x(\frac{1}{2}+n)}}{\sqrt{2\pi\sinh x}} \frac{(-1)^n\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{n!\,\Gamma(-n+\frac{1}{2})} F(\tfrac{1}{2},\tfrac{1}{2};-n+\tfrac{1}{2};\ -e^{-x}/(2\sinh x))$$

$$= \frac{\Gamma(n+\frac{1}{2})}{n!} \frac{e^{x(\frac{1}{2}+n)}}{\sqrt{2\pi\sinh x}} F(\tfrac{1}{2},\tfrac{1}{2};-n+\tfrac{1}{2};\ \frac{-e^{-x}}{2\sinh x}) .$$

If we now choose $t = \cos\theta$, (3.1) becomes

$$(3.9)\quad \sum_{n=o}^{\infty} P_n(\cos\theta)\, z^n = \left\{ 1 - z(e^{i\theta} + e^{-i\theta}) + z^2 \right\}^{-\frac{1}{2}}$$

$$= -\left\{z - e^{i\theta}\right\}^{-\frac{1}{2}} \left\{z - e^{-i\theta}\right\}^{-\frac{1}{2}} .$$

Again, the minus sign comes from choosing the branches of the functions $\{z - e^{i\theta}\}^{-\frac{1}{2}}$ and $\{z - e^{-i\theta}\}^{-\frac{1}{2}}$ that are equal to $-i\,e^{-\frac{1}{2}i\theta}$ and $-i\,e^{\frac{1}{2}i\theta}$ respectively at $z = 0$.

In this case, there are two singularities, $\alpha_1 = e^{i\theta}$ and $\alpha_2 = e^{-i\theta}$, both on the unit circle. Since the contribution of one singularity will clearly be the complex conjugate of the contribution of the other, we need only consider

$\alpha_1 = e^{i\theta}$, $g_1(z) = -\left\{ z - e^{-i\theta} \right\}^{-\frac{1}{2}}$. Then

$$(3.10) \cdot \quad g_1^{(r)}(z) = (-1)^{r+1} \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})} \left\{ z - e^{-i\theta} \right\}^{-(r+\frac{1}{2})} ,$$

and

$$(3.11) \quad g_1^{(r)}(e^{i\theta}) = (-1)^{r+1} \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})} \left\{ 2i \sin\theta \right\}^{-(r+\frac{1}{2})} .$$

Therefore, from (2.33), we have

$$(3.12) \quad P_n(\cos\theta) \sim 2\,\mathrm{Rl}\, \frac{(-1)^n}{n!} \sum_{r=0}^{\infty} \frac{\Gamma(n+\frac{1}{2}-r)\,\Gamma(r+\frac{1}{2})}{r!\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-r)} (-1)^{r+1} (-e^{i\theta})^{r-\frac{1}{2}-n} (2i\sin\theta)^{-(r+\frac{1}{2})}$$

$$= \frac{(-1)^n}{n!} \sum_{r=0}^{\infty} \frac{\left\{\Gamma(r+\frac{1}{2})\right\}^2 (2\sin\theta)^{-(r+\frac{1}{2})}}{\Gamma(r-n+\frac{1}{2})\; r!\; \Gamma(\frac{1}{2})} \, 2\,\mathrm{Rl}\left\{ e^{i\left[\theta(r-\frac{1}{2}-n)+\frac{\pi}{4}(1-2r)\right]}\right\}$$

$$= \frac{(-1)^n}{n!} \sqrt{\frac{2}{\pi\sin\theta}} \sum_{r=0}^{\infty} \frac{\left\{\Gamma(r+\frac{1}{2})\right\}^2 \cos\left\{\theta(n-r+\frac{1}{2}) + \frac{\pi}{4}(2r-1)\right\}}{\Gamma(r-n+\frac{1}{2})\; r!\; (2\sin\theta)^r}$$

As a second example we shall discuss the asymptotic behavior of the sequence $\{u_n\}$ whose generating function is given by

$$(3.13) \qquad \frac{e^{x + \frac{1}{2}x^2}}{\sqrt{1 - x}} = \sum_{n=0}^{\infty} \frac{u_n}{n!} x^n .$$

Letting $u_n = b_n / 2^n$, we have

$$(3.14) \qquad \sum_{n=0}^{\infty} \frac{b_n}{2^n} \frac{x^n}{n!} = \frac{e^{x + \frac{1}{2}x^2}}{\sqrt{1 - x}} = i \frac{e^{x + \frac{1}{2}x^2}}{\sqrt{x - 1}} .$$

The generating function has a branch point at $x = 1$. In the notation of chapter two,

$$(3.15) \qquad g(x) = i \; e^{x + \frac{1}{2} x^2}$$

and

$$(3.16) \qquad g^{(r)}(1) = \left[\frac{d^r}{dx^r} \left(i \; e^{x + \frac{1}{2}x^2}\right) \right]_{x=1}$$

$$= \left[\frac{d^r}{dx^r} \; i \; e^{(x+1) + \frac{1}{2}(x+1)^2} \right]_{x=0}$$

$$= i \, e^{\frac{3}{2}} \left[\frac{d^r}{dx^r} \; e^{\frac{1}{2}x^2 + 2x} \right]_{x=0} .$$

This derivative can be evaluated by the use of the Hermite polynomials. The generating function of these polynomials is

$$(3.17) \qquad e^{2tz - z^2} = \sum_{n=0}^{\infty} H_n(t) \frac{z^n}{n!} .$$

If $z = i x / \sqrt{2}$, $t = - i \sqrt{2}$, (3.17) becomes

$$(3.18) \quad e^{\frac{1}{2}x^2 + 2x} = \sum_{n=0}^{\infty} H_n(-\sqrt{2}\, i) \left(\frac{i}{\sqrt{2}}\right)^n \frac{x^n}{n!} .$$

Thus

$$(3.19) \quad \left[\frac{d^r}{dx^r} e^{\frac{1}{2}x^2 + 2x} \right]_{x=0} = \left(\frac{i}{\sqrt{2}}\right)^r H_r(-\sqrt{2}\, i) ,$$

and

$$(3.20) \quad g^{(r)}(1) = i\, e^{\frac{3}{2}} \left(\frac{i}{\sqrt{2}}\right)^r H_r(-\sqrt{2}\, i) .$$

The Hermite polynomials are given explicitly by

$$(3.21) \quad H_n(x) = n! \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^m \, (2x)^{n-2m}}{m! \, (n-2m)!} .$$

Hence

$$(3.22) \quad H_r(-\sqrt{2}\, i) = r! \, (-i)^r \sum_{m=0}^{\left[\frac{r}{2}\right]} \frac{(2\sqrt{2})^{r-2m}}{m! \, (r-2m)!} ,$$

and

$$(3.23) \quad H_0(-\sqrt{2}\, i) = 1 ,$$

$$H_1(-\sqrt{2}\, i) = -2\sqrt{2}\, i ,$$

$$H_2(-\sqrt{2}\, i) = -10 ,$$

$$H_3(-\sqrt{2}\, i) = 28\sqrt{2}\, i .$$

Therefore, from (2.33)

$$(3.24) \quad b_n \sim 2^n e^{\frac{3}{2}} \sum_{r=0}^{\infty} \left(-\frac{i}{\sqrt{2}}\right)^r \frac{\Gamma(n-r+\frac{1}{2})}{\Gamma(\frac{1}{2}-r)\, r!} H_r(-\sqrt{2}\, i) .$$

If we compute b_{15} and b_{25} using the first four terms of this expansion, i.e.,

$$(3.25) \qquad b_n \sim 2^n e^{\frac{3}{2}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} \left\{ 1 + \frac{2}{2n-1} + \frac{15}{2(2n-1)(2n-3)} + \frac{35}{(2n-1)(2n-3)(2n-5)} + \cdots \right\} ,$$

we obtain

$$(3.26) \qquad b_{15} = 2.997 \times 10^{16} ,$$

$$b_{25} = 2.7352 \times 10^{32} .$$

To determine the accuracy of this expansion, let us now establish a recurrence relationship for the sequence $\{b_n\}$ and find the exact values of these coefficients.

Differentiation of (3.13) with respect to x yields

$$(3.27) \qquad \frac{e^{x+\frac{1}{2}x^2}}{\sqrt{1-x}} (1+x) + \frac{e^{x+\frac{1}{2}x^2}}{2(1-x)^{3/2}} = \sum_{n=0}^{\infty} \frac{u_{n+1}}{n!} x^n$$

which simplifies to

$$(3.28) \qquad (\tfrac{3}{2} - x^2) \sum_{n=0}^{\infty} \frac{u_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{u_{n+1}}{n!} x^n (1-x) .$$

Hence we have the recurrence relationship

$$(3.29) \qquad (\tfrac{3}{2} + n) u_n - n(n-1) u_{n-2} = u_{n+1} ,$$

and the related equation

$$(3.30) \qquad b_{n+1} = (2n+3) b_n - 8 n(n-1) b_{n-2} .$$

By expanding the left hand side of (3.13), we obtain the initial values

$$(3.31) \qquad u_0 = 1 \ , \quad u_1 = \frac{3}{2} \ , \quad u_2 = \frac{15}{4} \ ,$$

and

$$(3.32) \qquad b_0 = 1 \ , \quad b_1 = 3 \ , \quad b_2 = 15 \ .$$

Successive use of (3.30), together with (3.32), yields the values given below.

$b_0 = 1$

$b_1 = 3$

$b_2 = 15$

$b_3 = 89$

$b_4 = 657$

$b_5 = 5\ 787$

$b_6 = 60\ 991$

$b_7 = 757\ 185$

$b_8 = 10\ 927\ 713$

$b_9 = 180\ 302\ 579$

$b_{10} = 3\ 350\ 215\ 599$

$b_{11} = 69\ 187\ 005\ 417$

$b_{12} = 1\ 571\ 008\ 865\ 905$

$b_{13} = 38\ 879\ 411\ 706\ 891$

$b_{14} = 1\ 041\ 157\ 556\ 739\ 423$

$b_{15} = 29\ 988\ 495\ 350\ 164\ 433$

$b_{16} = 924\ 302\ 934\ 887\ 849\ 409$

$b_{17} = 30\ 351\ 580\ 212\ 135\ 037\ 155$

$b_{18} = 1\ 057\ 753\ 501\ 967\ 038\ 568\ 527$

$b_{19} = 38\ 989\ 692\ 992\ 109\ 048\ 819\ 321$

$$b_{20} = 1\,515\,535\,489\,216\,069\,539\,936\,081$$

$$b_{21} = 61\,952\,455\,390\,311\,192\,968\,929\,403$$

$$b_{22} = 2\,656\,855\,124\,110\,517\,279\,568\,904\,575$$

$$b_{23} = 119\,270\,771\,665\,051\,719\,120\,134\,759\,649$$

$$b_{24} = 5\,593\,484\,272\,167\,554\,527\,748\,376\,999\,457$$

$$b_{25} = 273\,535\,025\,652\,473\,236\,608\,590\,944\,369\,107$$

A comparison of the results shows that b_{15} is determined to within a 0.06 percentage error, and b_{25} to within a 0.006 percentage error.

Example three is concerned with the following problem: If there be n straight lines in one plane, no three of which meet in a point, what is the number of groups of n of their points of intersection, in each of which no three points lie in one of the straight lines?

Let us denote the number of such groups by g_n. Robin Robinson [3.1] established the following recurrence relationship

$$(3.35) \qquad g_{n+1} = n\, g_n + \binom{n}{2} g_{n-2} \quad ,$$

where $g_1 = g_2 = 0 \quad , \quad g_3 = 1 \quad .$

He introduced u_n, where $g_n = \frac{1}{2}(n-1)!\; u_n$, and obtained the corresponding relationship

$$(3.36) \qquad u_{n+3} = u_{n+2} + \frac{u_n}{2n} \quad ,$$

with $u_1 = u_2 = 0 \quad , \quad u_3 = u_4 = u_5 = 1 \quad .$

Robinson proved that the number $b = \lim_{n \to \infty} u_n^2/n$ existed and consequently, that $g_n \sim \sqrt{\frac{\pi b}{2}}\; n^n\, e^{-n}$. By using the sequences $\left\{ u_{n+1}^2 - u_n^2 \right\}$ and $\left\{ u_n^2/(n - \frac{5}{4}) \right\}$, each of which converges to b, he was able to establish $b = 0.284,098,\ldots$, and expressed an interest in finding an exact expression for b.

The May, 1952 issue of the American Mathematical Monthly discussed the consequent articles that had been submitted. Various methods had been used to establish that Robinson's constant b is given by $b = 4\, e^{-\frac{3}{2}}/\pi$. The most common treatment involved the generating function

$$(3.37) \qquad f(x) = 2 \sum_{n=3}^{\infty} \frac{g_n}{n!} x^n = \sum_{n=3}^{\infty} \frac{u_n}{n} x^n .$$

An explicit expression for $f(x)$ was found from the recurrence relationship (3.36) to be

$$(3.38) \qquad f(x) = \frac{2 \, e^{-\frac{1}{4}x^2 - \frac{1}{2}x}}{\sqrt{1-x}} - 2 .$$

Three of the authors then used the following theorem: If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < 1$, and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, $|x| < 1$, where $\sum_{n=0}^{\infty} b_n$ diverges, and if $\lim_{n \to \infty} a_n / b_n = C$, then $\lim_{x \to 1} f(x)/g(x) = C$.

Applying this to the generating function $f(x)$, with $g(x) = (1-x)^{-\frac{1}{2}}$, they established that $\lim_{x \to 1} \frac{f(x)}{(1-x)^{-\frac{1}{2}}} = \sqrt{\pi b}$. But from (3.38) this limit also equals $2 e^{-\frac{3}{4}}$, whence $b = 4 e^{-\frac{3}{2}} / \pi$, and $g_n \sim \sqrt{2} \; n^n e^{-n-\frac{3}{4}}$.

One treatment employed contour integration and yielded, in addition to the exact value for b, an explicit expression for u_n.

Let us establish a more complete asymptotic expansion for g_n using the method of Darboux. We will also make use of the generating function

$$(3.39) \qquad f(x) = 2 \sum_{n=3}^{\infty} \frac{g_n}{n!} x^n .$$

By using (3.35), it is easily shown that

$$(3.40) \qquad (1 - x) f'(x) - x^2 = \tfrac{1}{2} x^2 f(x) .$$

Since $f(0) = 0$, (3.40) yields

$$(3.41) \quad f(x) = 2i \frac{e^{-\frac{1}{4}x^2 - \frac{1}{2}x}}{\sqrt{x-1}} - 2 .$$

The function $f(x)$ has a branch point at $x = 1$. In the notation of chapter two,

$$(3.42) \quad g(x) = 2i\, e^{-\frac{1}{4}x^2 - \frac{1}{2}x}$$

and

$$(3.43) \quad g^{(r)}(1) = 2i \left[\frac{d^r}{dx^r} e^{-\frac{1}{4}x^2 - \frac{1}{2}x} \right]_{x=1}$$

$$= 2i \left[\frac{d^r}{dx^r} e^{-\frac{1}{4}(x+1)^2 - \frac{1}{2}(x+1)} \right]_{x=0}$$

$$= 2i\, e^{-\frac{3}{4}} \left[\frac{d^r}{dx^r} e^{-\frac{1}{4}x^2 - x} \right]_{x=0} .$$

If $z = \frac{1}{2}x$, $t = -1$, (3.17) becomes

$$(3.44) \quad e^{-\frac{1}{4}x^2 - x} = \sum_{n=0}^{\infty} \frac{H_n(-1)}{2^n\, n!} x^n .$$

Thus

$$(3.45) \quad \left[\frac{d^r}{dx^r} (e^{-\frac{1}{4}x^2 - x}) \right]_{x=0} = \frac{H_r(-1)}{2^r} ,$$

and

$$(3.46) \quad g^{(r)}(1) = 2i\, e^{-\frac{3}{4}} \frac{H_r(-1)}{2^r} .$$

Using (3.21), we obtain

$$(3.47) \qquad H_r(-1) = r! \sum_{m=0}^{\left[\frac{r}{2}\right]} \frac{(-1)^{r-m}\, 2^{r-2m}}{m!\,(r-2m)!} ,$$

and

$$(3.48) \qquad \begin{aligned} H_0(-1) &= 1 \\ H_1(-1) &= -2 \\ H_2(-1) &= 2 \\ H_3(-1) &= 4 \end{aligned} .$$

Hence from (2.33)

$$(3.49) \qquad g_n \sim e^{-\frac{3}{4}} \sum_{r=0}^{\infty} \left(-\frac{1}{2}\right)^r \frac{\Gamma(n-r+\frac{1}{2})}{r!\,\Gamma(\frac{1}{2}-r)} H_r(-1) ,$$

$$(3.50) \qquad g_n \sim e^{-\frac{3}{4}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} \left\{ 1 - \frac{1}{2n-1} + \frac{3}{4(2n-1)(2n-3)} + \frac{5}{4(2n-1)(2n-3)(2n-5)} + \dots \right\} .$$

The dominant term of this expansion, together with Stirling's formula, yields the result previously discussed. i.e.,

$$(3.51) \qquad g_n \sim e^{-\frac{3}{4}} \sqrt{2}\; n^n\, e^{-n} .$$

Using first (3.51), then (3.50) to compute g_{15} and g_{25}, we obtain

$$(3.52) \qquad \begin{aligned} g_{15} &= 8.9 \times 10^{10} \\ g_{25} &= 8.2 \times 10^{23} \end{aligned}$$

and

$$(3.53)\qquad g_{15} = 8{,}625 \times 10^{7}$$

$$g_{25} = 806{,}127 \times 10^{18}$$

respectively.

The exact values of these coefficients are determined through successive use of (3.35) and are given below.

g_1	=	0
g_2	=	0
g_3	=	1
g_4	=	3
g_5	=	12
g_6	=	70
g_7	=	465
g_8	=	3 507
g_9	=	30 016
g_{10}	=	286 884
g_{11}	=	3 026 655
g_{12}	=	34 944 085
g_{13}	=	438 263 364
g_{14}	=	5 933 502 822
g_{15}	=	86 248 951 243
g_{16}	=	1 339 751 921 865
g_{17}	=	22 148 051 088 480
g_{18}	=	388 246 725 873 208
g_{19}	=	7 193 423 109 763 089
g_{20}	=	140 462 355 821 628 771
g_{21}	=	2 883 013 994 348 484 940

g_{22} = 62 053 912 734 368 432 430

g_{23} = 1 397 632 884 350 901 759 561

g_{24} = 32 874 958 880 640 907 159 723

g_{25} = 806 125 893 050 067 459 184 032

Comparing these results, we find g_{15} and g_{25} determined respectively to within 4 and 2 percentage errors by (3.51) and to within 0.001 and 0.0001 percentage errors by (3.50) .

We shall now establish an explicit expression for g_n . From (3.39) and (3.41) ,

$$(3.54) \qquad g_n = \left[\frac{d^n}{dx^n} \left\{ i \, \frac{e^{-\frac{1}{4}x^2 - \frac{1}{2}x}}{\sqrt{x-1}} - 1 \right\} \right]_{x=0}$$

$$= i \sum_{r=o}^{n} \binom{n}{r} \left[\frac{d^{n-r}}{dx^{n-r}} \, e^{-\frac{1}{4}x^2 - \frac{1}{2}x} \right]_{x=0} \left[\frac{d^r}{dx^r} (x-1)^{-\frac{1}{2}} \right]_{x=0} .$$

If $z = \frac{1}{2}x$, $t = -\frac{1}{2}$, (3.17) becomes

$$(3.55) \qquad e^{-\frac{1}{4}x^2 - \frac{1}{2}x} = \sum_{n=0}^{\infty} \frac{H_n(-\frac{1}{2})}{2^n \; n!} x^n .$$

Thus

$$(3.56) \qquad \left[\frac{d^{n-r}}{dx^{n-r}} \, e^{-\frac{1}{4}x^2 - \frac{1}{2}x} \right]_{x=0} = \frac{H_{n-r}(-\frac{1}{2})}{2^{n-r}} ,$$

where, from (3.21)

$$(3.57) \qquad H_{n-r}(-\tfrac{1}{2}) = (n-r)! \sum_{m=o}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^{n-r-m}}{m!\ (n-r-2m)!} .$$

Since

$$(3.58) \qquad \left[\frac{d^r}{dx^r} (x-1)^{-\frac{1}{2}} \right]_{x=0} = - \frac{i\ (2r)!}{2^{2r}\ r!} ,$$

we have

$$(3.59) \qquad g_n = \frac{n!}{2^n} \sum_{r=o}^{n} \binom{2r}{r} \frac{H_{n-r}(-\frac{1}{2})}{(n-r)!\ 2^r}$$

$$= \frac{n!}{2^n} \sum_{r=o}^{n} \binom{2r}{r} \frac{1}{2^r} \sum_{m=o}^{\left[\frac{n-r}{2}\right]} \frac{(-1)^{n-r-m}}{m!\ (n-r-2m)!} .$$

Example four is concerned with the number of distinct terms in a symmetrical or partially symmetrical determinant.

In dealing with this problem, Professor A. Cayley [3.2] represented the terms of a determinant as duads, and the determinant as a bicolumn. For instance,

$$(3.60)\qquad \left\{\begin{matrix} aa \\ bb \\ pp' \\ qq' \end{matrix}\right\} \quad \text{represented the determinant} \quad \begin{vmatrix} aa & ab & ap' & aq' \\ ba & bb & bp' & bq' \\ pa & pb & pp' & pq' \\ qa & qb & qp' & qq' \end{vmatrix} ,$$

where the duads had the property that $sr = rs$.

Thus a determinant whose bicolumn contains such symbols as pp' and qq' , (each letter $p, q, \ldots$ being distinct from each letter $p', q', \ldots$) is partially symmetrical, while a determinant such as $\left\{\begin{matrix} aa \\ bb \\ cc \end{matrix}\right\}$ is wholly symmetrical.

Cayley defined a determinant (m,n) to be one whose bicolumn had m rows $aa, bb, \ldots$ and n rows $pp', qq', \ldots$, and denoted the number of distinct terms in the developed expression of such a determinant by $\phi(m,n)$.

He established the recurrence relationships

$$(3.61)\qquad \phi(m,n) = m\,\phi(m-1,n) + n\,\phi(m,n-1) \quad , \quad n \geq 1$$

$$(3.62)\qquad \phi(m,0) = \phi(m-1,0) + (m-1)\,\phi(m-2,0) + \tfrac{1}{2}(m-1)(m-2)\,\phi(m-3,1)$$

and used these equations together with the initial values

$\phi(0,0) = 1$, $\phi(1,0) = \phi(0,1) = 1$, $\phi(2,0) = \phi(1,1) = \phi(0,2) = 2$,

to calculate values of $\phi(m,n)$ in the following order

$$\phi(0,0)$$
$$\phi(1,0) \quad \phi(0,1)$$
$$\phi(2,0) \quad \phi(1,1) \quad \phi(0,2)$$
$$\phi(3,0) \quad \phi(2,1) \quad \phi(1,2) \quad \phi(0,3)$$
$$\phi(4,0) \quad \phi(3,1) \quad \phi(2,2) \quad \phi(1,3) \quad \phi(0,4)$$
$$\phi(5,0) \quad \phi(4,1) \quad \phi(3,2) \quad \phi(2,3) \quad \phi(1,4) \quad \phi(0,5)$$
$$\phi(6,0) \quad \phi(5,1) \quad \phi(4,2) \quad \phi(3,3) \quad \phi(2,4) \quad \phi(1,5) \quad \phi(0,6)$$

The values obtained were

$$1$$
$$1 \quad 1$$
$$2 \quad 2 \quad 2$$
$$5 \quad 6 \quad 6 \quad 6$$
$$17 \quad 23 \quad 24 \quad 24 \quad 24$$
$$73 \quad 109 \quad 118 \quad 120 \quad 120 \quad 120$$
$$388 \quad 618 \quad 690 \quad 714 \quad 720 \quad 720 \quad 720$$

Introducing

$$(3.63)\qquad u = \phi(0,0) + \phi(1,0)\frac{x}{1} + \phi(2,0)\frac{x^2}{2} + \ldots + \phi(m,0)\frac{x^m}{m!} + \ldots ,$$

Cayley used (3.61) and (3.62) to show that

$$(3.64)\qquad 2\frac{du}{dx} - u - xu = \frac{u}{1-x} .$$

Hence he obtained

$$(3.65)\qquad u = \frac{e^{\frac{1}{2}x + \frac{1}{4}x^2}}{\sqrt{1-x}}$$

and

$$(3.66)\qquad \phi(m,0) = m! \left\{ \text{coefficient of } x^m \text{ in } \frac{e^{\frac{1}{2}x + \frac{1}{4}x^2}}{\sqrt{1-x}} \right\} .$$

Further results were

$$(3.67)\qquad \phi(m,1) = m! \left\{ \text{coefficient of } x^m \text{ in } e^{\frac{1}{2}x + \frac{1}{4}x^2}/(1-x)^{\frac{3}{2}} \right\}$$

$$\phi(m,2) = m! \left\{ \text{coefficient of } x^m \text{ in } 2!\; e^{\frac{1}{2}x + \frac{1}{4}x^2}/(1-x)^{\frac{5}{2}} \right\}$$

$$\phi(m,3) = m! \left\{ \text{coefficient of } x^m \text{ in } 3!\; e^{\frac{1}{2}x + \frac{1}{4}x^2}/(1-x)^{\frac{7}{2}} \right\}$$

$$\ldots$$

From (3.64) , Cayley then derived the following relation for finding the number of terms in a symmetrical determinant of order n

$$(3.68)\qquad \phi(n,0) = n\,\phi(n-1,0) - \tfrac{1}{2}(n-1)(n-2)\,\phi(n-3,0) .$$

We shall now investigate the asymptotic behavior of $\phi(m,n)$, for large m . Using (3.61) , we find

$$(3.69) \quad \sum_{m,n=1}^{\infty} \frac{\phi(m,n)}{m!\;\; n!} x^m y^n = \frac{y}{y-1} + \frac{y}{1-x-y} \frac{e^{\frac{1}{2}x + \frac{1}{4}x^2}}{\sqrt{1-x}}$$

and

$$(3.70) \quad \sum_{m,n=o}^{\infty} \frac{\phi(m,n)}{m!\;\; n!} x^m y^n = \frac{1-x}{1-x-y} \cdot \frac{e^{\frac{1}{2}x + \frac{1}{4}x^2}}{\sqrt{1-x}} .$$

Hence

$$(3.71) \quad \phi(m,n) = \left[\frac{\partial^m}{\partial x^m} \left[\frac{\partial^n}{\partial y^n} \left\{ \frac{1-x}{1-x-y} \frac{e^{\frac{1}{2}x + \frac{1}{4}x^2}}{\sqrt{1-x}} \right\} \right]_{y=0} \right]_{x=0}$$

$$= \left[\frac{d^m}{dx^m} \; n! \; \frac{e^{\frac{1}{2}x + \frac{1}{4}x^2}}{(1-x)^{n+\frac{1}{2}}} \right]_{x=0}$$

and

$$(3.72) \quad \sum_{m=o}^{\infty} \frac{\phi(m,n)}{m!} x^m = n! \frac{e^{\frac{1}{2}x + \frac{1}{4}x^2}}{(1-x)^{n+\frac{1}{2}}} = i\,(-1)^n\, n! \frac{e^{\frac{1}{2}x + \frac{1}{4}x^2}}{(x-1)^{n+\frac{1}{2}}} .$$

This generating function has a branch point at $x=1$. In the notation of chapter two,

$$(3.73) \quad g(x) = i\,(-1)^n \; n! \; e^{\frac{1}{2}x + \frac{1}{4}x^2}$$

and

$$(3.74) \quad g^{(r)}(1) = i\,(-1)^n \; n! \; e^{\frac{3}{4}} \left[\frac{d^r}{dx^r} \; e^{\frac{1}{4}x^2 + x} \right]_{x=0}$$

If we make the substitution $z = \frac{1}{2} i x$, $t = - i$, (3.17) yields

$$(3.75) \qquad \left[\frac{d^r}{dx^r} \left(e^{\frac{1}{4}x^2 + x}\right) \right]_{x=0} = H_r(-\,i) \left(\frac{i}{2}\right)^r$$

whence

$$(3.76) \qquad g^{(r)}(1) = i\,(-1)^n\, n!\; e^{\frac{3}{4}} \left(\frac{i}{2}\right)^r H_r(-\,i) \quad .$$

As a result of (3.21) ,

$$(3.77) \qquad H_r(-\,i) = (-\,i)^r\; r! \sum_{s=0}^{\left[\frac{r}{2}\right]} \frac{2^{r-2s}}{s!\,(r-2s)!}$$

and

$$(3.78) \qquad \begin{aligned} H_0(-\,i) &= 1 \\ H_1(-\,i) &= -2i \\ H_2(-\,i) &= -6 \\ H_3(-\,i) &= 20i \quad . \end{aligned}$$

Hence, from (2.33)

$$(3.79) \qquad \phi(m,n) \sim n!\; e^{\frac{3}{4}} \sum_{r=0}^{\infty} \left(-\frac{i}{2}\right)^r \frac{\Gamma(m-r+n+\frac{1}{2})}{r!\,\Gamma(n+\frac{1}{2}-r)}\, H_r(-\,i)$$

To obtain an explicit expression for $\phi(m,n)$, we first apply the rule of Leibniz to (3.71) . i.e.,

$$(3.80) \qquad \phi(m,n) = n! \sum_{r=0}^{m} \binom{m}{r} \left[\frac{d^r}{dx^r} (1-x)^{-n-\frac{1}{2}} \right]_{x=0} \left[\frac{d^{m-r}}{dx^{m-r}} \left(e^{\frac{1}{2}x + \frac{1}{4}x^2}\right) \right]_{x=0}$$

$$= n! \sum_{r=0}^{m} \binom{m}{r} \frac{\Gamma(n+r+\frac{1}{2})}{\Gamma(n+\frac{1}{2})} \left[\frac{d^{m-r}}{dx^{m-r}} \left(e^{\frac{1}{2}x + \frac{1}{4}x^2}\right) \right]_{x=0} .$$

If $z = \frac{1}{2} ix$, $t = -\frac{1}{2} i$, (3.17) becomes

$$(3.81) \qquad e^{\frac{1}{2}x + \frac{1}{4}x^2} = \sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n H_n\left(-\frac{i}{2}\right) \frac{x^n}{n!} .$$

Hence, using (3.21) ,

$$(3.82) \qquad \left[\frac{d^{m-r}}{dx^{m-r}} \left(e^{\frac{1}{2}x + \frac{1}{4}x^2}\right) \right]_{x=0} = \left(\frac{i}{2}\right)^{m-r} H_{m-r}\left(-\frac{i}{2}\right)$$

$$= \frac{(m-r)!}{2^{m-r}} \sum_{s=0}^{\left[\frac{m-r}{2}\right]} \frac{1}{s!\ (m-r-2s)!} ,$$

and

$$(3.83) \qquad \phi(m,n) = \frac{n!\ m!}{2^m} \sum_{r=0}^{m} \frac{\Gamma(n+r+\frac{1}{2})\ 2^r}{\Gamma(n+\frac{1}{2})\ \ r!} \sum_{s=0}^{\left[\frac{m-r}{2}\right]} \frac{1}{s!\ (m-r-2s)!} .$$

Let us now derive a second explicit expression for $\phi(m,n)$.
Substituting $x = u/n$ in (3.72) , we have

$$(3.84) \qquad \phi(m,n) = n!\ n^m \left[\frac{d^m}{du^m} \frac{e^{\frac{1}{2}\frac{u}{n} + \frac{1}{4}\left(\frac{u}{n}\right)^2}}{(1-\frac{u}{n})^{n+\frac{1}{2}}} \right]_{u=0} .$$

However,

$$(3.85) \qquad \frac{e^{\frac{1}{2}\frac{u}{n} + \frac{1}{4}\left(\frac{u}{n}\right)^2}}{(1-\frac{u}{n})^{n+\frac{1}{2}}} = e^{u + \frac{1}{n}(u + \frac{u^2}{2}) + \frac{1}{n^2}\left(\frac{u^2}{2} + \frac{u^3}{3}\right) + \sum_{r=3}^{\infty} \frac{1}{n^r}\left(\frac{u^r}{2r} + \frac{u^{r+1}}{r+1}\right)}$$

$$= e^u \left\{ 1 + \sum_{r=1}^{\infty} A_r(u) \cdot \frac{1}{n^r} \right\}$$

where

$$(3.86)\qquad \begin{aligned} A_1(u) &= u + \tfrac{1}{2}u^2 \\ A_2(u) &= u^2 + \tfrac{5}{6}u^3 + \tfrac{1}{8}u^4 \\ A_3(u) &= \tfrac{5}{6}u^3 + \tfrac{13}{12}u^4 + \tfrac{7}{24}u^5 + \tfrac{1}{48}u^6 \\ A_4(u) &= \tfrac{17}{24}u^4 + \tfrac{6}{5}u^5 + \tfrac{17}{36}u^6 + \tfrac{1}{16}u^7 + \tfrac{1}{384}u^8 \end{aligned}$$

It can be shown that $A_r(u)$ is a polynomial in u in which the least power of u is r, and the degree is $2r$.

Thus

$$(3.87)\qquad \phi(m,n) = n!\,n^m \left[\frac{d^m}{du^m} e^u \left\{ 1 + \sum_{r=1}^{\infty} A_r(u) \frac{1}{n^r} \right\} \right]_{u=0}$$

$$= n!\,n^m \left\{ 1 + \sum_{r=1}^{m} \left[\frac{d^m}{du^m} e^u A_r(u) \right]_{u=0} \cdot \frac{1}{n^r} \right\} .$$

Since $\left[\dfrac{d^m}{du^m} (e^u u^s) \right]_{u=0} = \dfrac{m!}{(m-s)!}$, we obtain

$$(3.88)\qquad \phi(m,n) = n!\,n^m \left\{ 1 + \frac{B_1}{n} + \frac{B_2}{n^2} + \dots + \frac{B_m}{n^m} \right\}$$

where

$$(3.89)\qquad \begin{aligned} B_1 &= m + \tfrac{1}{2}m(m-1) \\ B_2 &= m(m-1) + \tfrac{5}{6}m(m-1)(m-2) + \tfrac{1}{8}m(m-1)(m-2)(m-3) \\ B_3 &= \tfrac{5}{6}m(m-1)(m-2) + \tfrac{13}{12}m(m-1)(m-2)(m-3) + \tfrac{7}{24}m(m-1)\dots(m-4) \\ &\quad + \tfrac{1}{48}m(m-1)\dots(m-5) \\ B_4 &= \tfrac{17}{24}m(m-1)\dots(m-3) + \tfrac{6}{5}m(m-1)\dots(m-4) + \tfrac{17}{36}m(m-1)\dots(m-5) \\ &\quad + \tfrac{1}{16}m(m-1)\dots(m-6) + \tfrac{1}{384}m(m-1)\dots(m-7) \end{aligned}$$

For example,

$$(3.90) \qquad \phi(3,n) = n!\, n^3 \left\{ 1 + \frac{6}{n} + \frac{11}{n^2} + \frac{5}{n^3} \right\}$$

$$\phi(4,n) = n!\, n^4 \left\{ 1 + \frac{10}{n} + \frac{35}{n^2} + \frac{46}{n^3} + \frac{17}{n^4} \right\} .$$

Example five deals with the number $f(n)$ of representations of an integer n as an ordered product of factors greater than 1. The asymptotic behavior of $f(n)$ for squarefree n was determined by Abe Sklar [3.3]. He defined $h(r) = f(S_r)$, where S_r represented any squarefree integer with r prime factors, and applied the Euler-MacLaurin formula to the result

$$(3.91) \qquad h(r) = \tfrac{1}{2} \sum_{n=o}^{\infty} n^r\, 2^{-n} ,$$

to obtain the following theorem:

(a) For every positive integral r ,

$$(3.92) \qquad h(r) = 2^{-1}\, r!\, (\log 2)^{-r-1} \left\{ 1 + R_r \right\} ,$$

and the remainder R_r can be expressed in the form:

$$(3.93) \qquad 2 \sum_{n=1} (\cos \theta_n)^{r+1} \cos \left\{ (r-1)\, \theta_n \right\} ,$$

where θ_n is defined by:

$$(3.94) \qquad \cos \theta_n = \frac{\log 2}{2\pi n} \left\{ 1 + \left(\frac{\log 2}{2\pi n} \right)^2 \right\}^{-\frac{1}{2}} .$$

(b) For every positive integral r ,

$$(3.95) \qquad \left| R_r \right| \leq 2\, \zeta(r+1) \left(\frac{\log 2}{2\pi} \right)^{r+1}$$

where $\zeta(r+1)$ is the Riemann zeta function.

Let us first develop an explicit expression for $h(r)$.

Using (3.91) , we find

$$(3.96)\qquad \sum_{r=0}^{\infty} h(r)\,\frac{x^r}{r!} = \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\, e^{nx} = \frac{\frac{1}{2}}{1-\frac{1}{2}e^x} .$$

Thus

$$(3.97)\qquad h(r) = \frac{1}{2}\left[\frac{d^r}{dx^r}\left(1-\frac{1}{2}e^x\right)^{-1}\right]_{x=0}$$

$$= \frac{1}{2}\left[\frac{d^r}{du^r}\left(1-e^u\right)^{-1}\right]_{u=-\log 2} .$$

This derivative can be evaluated using the Bernoulli numbers. The generating function of these numbers is given by

$$(3.98)\qquad \frac{u}{e^u-1} = \sum_{m=0}^{\infty} \frac{B_m}{m!}\, u^m$$

whence

$$(3.99)\qquad \frac{1}{e^u-1} = \frac{1}{u} + \sum_{m=1}^{\infty} \frac{B_m}{m!}\, u^{m-1} .$$

Therefore,

$$(3.100)\qquad \frac{d^r}{du^r}\left(e^u-1\right)^{-1} = \frac{(-1)^r\, r!}{u^{r+1}} + \sum_{m=r+1}^{\infty} \frac{B_m}{m}\,\frac{u^{m-r-1}}{(m-r-1)!}$$

$$= \frac{(-1)^r\, r!}{u^{r+1}} + \sum_{k=0}^{\infty} \frac{B_{k+r+1}}{k+r+1}\,\frac{u^k}{k!} .$$

and

$$(3.101)\quad h(r) = \tfrac{1}{2}\left\{ \frac{r!}{(\log 2)^{r+1}} - \sum_{k=o}^{\infty} \frac{B_{k+r+1}}{k+r+1} \frac{(-\log 2)^k}{k!} \right\}$$

$$= \tfrac{1}{2} \frac{r!}{(\log 2)^{r+1}} \left\{ 1 - \frac{B_{r+1}}{r+1} \frac{(\log 2)^{r+1}}{r!} \right.$$

$$\left. + \frac{B_{r+2}}{r+2} \frac{(\log 2)^{r+2}}{r!\ 1!} + \ldots \right\}$$

where the Bernoulli numbers are given explicitly by

$$(3.102)\quad B_n = \sum_{m=1}^{n} \sum_{r=1}^{m} \frac{(-1)^r}{m+1} \binom{m}{r} r^n \quad .$$

We shall now derive the asymptotic behavior of $h(r)$ from (3.101). The following identity will be useful.

$$(3.103)\quad \frac{B_{2n}}{(2n)!} = \frac{2\,(-1)^{n+1}}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \quad .$$

Let us first suppose that $r = 2s - 1$. Then

$$(3.104)\quad B_{r+2} = B_{r+4} = \ldots = B_{2n+1} = 0 \quad , \qquad n \geqslant 1$$

and, using (3.103) ,

$$(3.105)\quad \left| \frac{B_{r+1}}{r+1} \frac{(\log 2)^{r+1}}{r!} \right| = 2 \left(\frac{\log 2}{2\pi}\right)^{r+1} \sum_{k=1}^{\infty} \frac{1}{k^{r+1}} \quad .$$

But

$$(3.106)\quad \sum_{k=1}^{\infty} \frac{1}{k^{r+1}} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad ,$$

and

(3.107) $$\left(\frac{\log 2}{2\pi}\right)^{r+1} < 0.2\, e^{-2r} .$$

Thus

(3.108) $$\left| \frac{B_{r+1}}{r+1} \frac{(\log 2)^{r+1}}{r!} \right| < e^{-2r} \sim 0 .$$

Clearly, all the terms involving even Bernoulli numbers will be asymptotic to zero.

A similar result follows if $r = 2s$.

Hence, we have shown

(3.109) $$h(r) \sim \tfrac{1}{2}\, r!\, (\log 2)^{-r-1} .$$

The above result can be obtained very easily using the method of Darboux. From (3.96) , we find that the generating function of the sequence $\{ h(r) \}$ has singularities at $x = \log 2 + 2n\pi i$. Thus the nearest singularity is at $x = \log 2$. If we write (3.96) in the form

(3.110) $$\sum_{r=0}^{\infty} h(r) \frac{x^r}{r!} = \frac{1}{x - \log 2} \quad \frac{x - \log 2}{2 - e^x}$$

we see that the singularity at $x = \log 2$ is a pole of order one and that, in the notation of chapter two,

(3.111) $$g(x) = (x - \log 2)/(2 - e^x) .$$

In this case, $g(x)$ has a removable singularity at $x = \log 2$, and can be made regular there by defining

(3.112) $$g(\log 2) = \lim_{x \to \log 2} \frac{x - \log 2}{2 - e^x} = -\tfrac{1}{2} .$$

Since we are dealing with a pole of order one, (2.33) yields a one-term asymptotic expansion for $h(r)$, i.e.,

(3.113) $$h(r) \sim \tfrac{1}{2}\, r!\, (\log 2)^{-r-1} .$$

The study of the asymptotic behavior of sequences occurs in many branches of mathematics. The solution of this problem by the method of Darboux requires the existence of a generating function with a finite number of singularities of a particular type. Thus a glance at the generating function determines whether or not the method is applicable. Examples three and five suggest that the method of Darboux is not as well known as it should be.

It is obvious of course, that the method of Darboux does not solve the most general case of the asymptotic behavior of sequences. For example, if the generating function is an integral function, the method fails. However, several other methods have been developed for such generating functions. One would have to say that the general problem of the asymptotic behavior of arbitrary sequences is still unsolved.

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